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Some aspects of time-dependent motion of a stratified rotating fluid

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Time-dependent motion of a rotating stratified fluid is analyzed within the quasi-geostrophic approximation. A few examples of mechanically driven flow are analyzed. It is found that the motion is characterized by the ratio B of the stability frequency and the Coriolis parameter. Thus the ratio of the horizontal and vertical characteristic scale is in general $O(B)$. In particular the decay process caused by a horizontal boundary will penetrate a distance $B^{-1}L$ into the fluid, L denoting the horizontal scale of the motion.

1. Introduction

We consider here some aspects of small amplitude time-dependent motion of a stratified rotating fluid. We will assume that the time scale of the process under consideration is large compared to the rotation time but small compared to the time that is required for diffusion to penetrate through the interior of the system. An example of this type of motion that has been given some attention in the literature is the spin up process that occurs as a response to a small change in the rate of rotation of the container that encloses the fluid. Greenspan & Howard (1963) worked out this problem in detail in the case of a homogeneous fluid. The spin up of a stratified fluid has been treated by Holton (1965) and Pedlosky (1967) with strikingly different results. It is shown here that neither treatment is correct. Holton obtained a solution with the right qualitative features but used an incorrect boundary condition on the vertical wall of his cylindrical container, disregarding the fact that the boundary layer on this wall when thermally insulating is unable to transport an appreciable amount of fluid. Pedlosky, on the other hand, realized this but drew the false conclusion that the boundary layer on the bottom does not exist and consequently that 'the interior spins up by a strictly diffusive process'.

Besides the 'spin up' problem, a few other illustrative examples of time-dependent flow will be analyzed.

2. The quasigeostrophic approximation

2.1 *The approximate equations*

In this section we will derive a system of approximate equations governing small amplitude time-dependent motion in the parameter range typical for geophysical phenomena. In this connexion we will not specify any complete initial or boundary conditions. This means that we are forced to base the derivation on requirements on the scales of the interior fields rather than on scales imposed on the system from the exterior. However, conditions in terms of exterior quantities may always be formulated, when relations between scales imposed through the boundary conditions and the interior scales have been found.

Within the Boussinesq approximation the equations governing motions of an inhomogeneous fluid in a frame of reference rotating with angular velocity Ω may be written

$$\rho_0\left(\frac{d\mathbf{v}^*}{dt^*}+2\Omega\mathbf{k}\times\mathbf{v}^*\right) = -\nabla^* p_t^* - \rho_t^* g\mathbf{k} + \rho_0 \nu \nabla^{*2}\mathbf{v}^*, \qquad (2.1a)$$

$$d\rho_t^*/dt^* = \kappa\nabla^{*2}\rho_t^*, \qquad (2.1b)$$

$$\nabla^*.\mathbf{v}^* = 0, \qquad (2.1c)$$

where $\nabla^* = \left(\frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*}, \frac{\partial}{\partial z^*}\right)$, $\frac{d}{dt^*} = \frac{\partial}{\partial t^*} + u^*\frac{\partial}{\partial x^*} + v^*\frac{\partial}{\partial y^*} + w^*\frac{\partial}{\partial z^*}$,

$\mathbf{v}^*$ is the velocity vector with components (u^*, v^*, w^*) in the Cartesian coordinate system (x^*, y^*, z^*), ν and κ are the diffusivities of momentum and density, $\mathbf{k}$ is a vector with components $(0, 0, 1)$, p_t^* is the pressure, ρ_t^* is the density and ρ_0 is a constant density such that

$$|\rho_t^* - \rho_0| \ll \rho_t^*.$$

In deriving (2.1) we have made two assumptions not generally included in the Boussinesq approximation. Thus we have neglected the curvature of the geopotential surfaces (including the potential of the centrifugal force) and assumed that the gravitational acceleration $-g\mathbf{k}$ is antiparallel to the rotation vector $\Omega\mathbf{k}$. (It may be pointed out that although this last assumption is usually not valid in geophysical applications, we obtain essentially the same set of equations without this assumption, provided the depth of the fluid is much smaller than the horizontal scale of motion, which is often the case in geophysical situations.)

Let us split up the density and pressure field into three parts according to

$$\rho_t^* = \rho_0 + \rho_s^*(z^*, t^*) + \rho^*(x^*, y^*, z^*, t^*), \qquad (2.2a)$$

$$p_t^* = p_s^*(z^*, t^*) + p^*(x^*, y^*, z^*, t^*), \qquad (2.2b)$$

where ρ_s^* and p_s^* are defined by

$$\partial\rho_s^*/\partial t^* = \kappa\,\partial^2\rho_s^*/\partial z^{*2}, \qquad (2.3)$$

$$0 = -\nabla^* p_s^* - (\rho_0 + \rho_s^*)\, g\mathbf{k}, \qquad (2.4)$$

together with initial and boundary conditions on ρ_s^* depending only on z^* and t^* (to be discussed in § 2.2).

We observe that the definition of ρ_s^* guarantees that ρ_s^* is independent of x^* and y^* for all times, which is required by (2.4). Let us now substitute (2.2) into (2.1) utilizing (2.3) and (2.4). We obtain

$$\rho_0\left(\frac{d\mathbf{v}^*}{dt^*}+2\Omega\mathbf{k}\times\mathbf{v}^*\right) = -\nabla^* p^* - \rho^* g\mathbf{k} + \rho_0 \nu \nabla^{*2}\mathbf{v}^*, \qquad (2.5a)$$

$$\frac{d\rho^*}{dt^*} + w^*\frac{\partial\rho_s^*}{\partial z^*} = \kappa\nabla^{*2}\rho^*, \qquad (2.5b)$$

$$\nabla^*.\mathbf{v}^* = 0. \qquad (2.5c)$$

We will not consider boundary conditions involving the pressure. Accordingly (2.4) will not be needed since (2.5) and (2.3) are independent of p_s^*.

Observe, however, that ρ_s^* has to be adopted as a new dependent variable. Equations (2.3) and (2.5) will now be non-dimensionalized using the following transformations

$$(x^*, y^*, z^*) = L(x, y, z), \tag{2.6a}$$

$$t^* = \tau t, \tag{2.6b}$$

$$(u^*, v^*, w^*) = U(u, v, w), \tag{2.6c}$$

$$p^* = Pp = 2\Omega U \rho_0 L p, \tag{2.6d}$$

$$\rho^* = Q\rho = 2\Omega U \rho_0 g^{-1} \rho, \tag{2.6e}$$

$$\rho_s^* = Q_s \rho_s. \tag{2.6f}$$

The choice of P and Q is motivated by the expectancy that the pressure gradient together with the gravitation and Coriolis forces should dominate the momentum equations.

We obtain in component form

$$\delta \frac{\partial u}{\partial t} + R\mathbf{v}.\nabla u - v = -\frac{\partial}{\partial x} p + E\nabla^2 u, \tag{2.7a}$$

$$\delta \frac{\partial v}{\partial t} + R\mathbf{v}.\nabla v + u = -\frac{\partial}{\partial y} p + E\nabla^2 v, \tag{2.7b}$$

$$\delta \frac{\partial w}{\partial t} + R\mathbf{v}.\nabla w + \rho = -\frac{\partial}{\partial z} p + E\nabla^2 w, \tag{2.7c}$$

$$\delta \frac{\partial \rho}{\partial t} + R\mathbf{v}.\nabla \rho + B^2 \frac{\partial \rho_s}{\partial z} w = \sigma^{-1} E \nabla^2 \rho, \tag{2.7d}$$

$$\nabla.\mathbf{v} = 0, \tag{2.7e}$$

$$\delta \frac{\partial \rho_s}{\partial t} = \sigma^{-1} E \frac{\partial^2 \rho_s}{\partial z^2}, \tag{2.7f}$$

where

$$\delta = 1/\tau 2\Omega, \tag{2.8a}$$

$$R = U/2\Omega L, \tag{2.8b}$$

$$E = \nu/2\Omega L^2, \tag{2.8c}$$

$$\sigma = \nu/\kappa, \tag{2.8d}$$

$$B = N/2\Omega = (Q_s g/\rho_0 L)^{\frac{1}{2}}/2\Omega. \tag{2.8e}$$

We want to study quasigeostrophic motion with a dimensional time scale, large compared to the rotation time but still small compared to the diffusive time scales L^2/ν and L^2/κ. If

$$\sigma \geqslant O(1), \tag{2.9a}$$

the corresponding restriction on δ becomes

$$1 \gg \delta \gg E. \tag{2.9b}$$

The non-linear terms may always be neglected compared to the local time derivatives if

$$R \ll \delta. \tag{2.9c}$$

When the process under consideration is axisymmetric, the influence of the non-linear terms in (2.7) is reduced since the main part of the velocity becomes perpendicular to the momentum and density gradients. It turns out that in this case (2.9c) may be replaced by the much weaker condition

$$R \ll 1. \tag{2.9d}$$

The parameter B is assumed to satisfy

$$B = O(1) \tag{2.9e}$$

when compared with the small parameter δ. From the definition of Q given by (2.6e) combined with (2.8) we can derive

$$Q/Q_s = B^{-2}R, \tag{2.10}$$

or in view of (2.9) and (2.6e,f)

$$\rho^* \ll \rho_s^*. \tag{2.11}$$

Thus in the régime defined by (2.9) the field of density anomaly is dominated by ρ_s^*, and ρ^* may be considered as a perturbation on the 'basic stratification' ρ_s^*.

The relative importance of this basic stratification and the mean rotation of the system is expressed by the ratio B of the stability frequency, $N = \sqrt{(Q_s g/\rho_0 L)}$, and the Coriolis parameter, 2Ω. In the interior of the system, where the scales introduced in (2.6) are representative, the magnitude of an individual term in (2.7) is measured by its coefficient. Thus we may take advantage of conditions (2.9) through an expansion, valid in the interior, according to

$$u = u^0 + u'\delta \ldots, \tag{2.12a}$$

$$v = v^0 + v'\delta \ldots, \tag{2.12b}$$

$$w = w^0 + w'\delta \ldots, \tag{2.12c}$$

$$p = p^0 + p'\delta \ldots, \tag{2.12d}$$

$$\rho = \rho^0 + \rho'\delta \ldots, \tag{2.12e}$$

$$\rho_s = \rho_s^0 + \rho_s'\delta \ldots. \tag{2.12f}$$

When (2.12) is substituted in (2.7) utilizing (2.9a–e) we obtain the zero-order equations

$$-v^0 = -\partial p^0/\partial x, \tag{2.13a}$$

$$+u^0 = -\partial p^0/\partial y, \tag{2.13b}$$

$$\rho^0 = -\partial p^0/\partial z, \tag{2.13c}$$

$$B^2 w^0 \partial \rho_s^0/\partial z = 0, \tag{2.13d}$$

$$\partial u^0/\partial x + \partial v^0/\partial y + \partial w^0/\partial z = 0, \tag{2.13e}$$

$$\partial \rho_s^0/\partial t = 0 \tag{2.13f}$$

and the first-order equations

$$\frac{\partial u^0}{\partial t} - v' = -\frac{\partial p'}{\partial x}, \tag{2.14a}$$

$$\frac{\partial v^0}{\partial t} + u' = -\frac{\partial p'}{\partial y}, \tag{2.14b}$$

$$\frac{\partial w^0}{\partial t}+\rho' = -\frac{\partial p'}{\partial z}, \tag{2.14c}$$

$$\frac{\partial \rho^0}{\partial t}+B^2\frac{\partial \rho_s^0}{\partial z}w'+B^2\frac{\partial \rho_s'}{\partial z}w^0 = 0, \tag{1.24d}$$

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z} = 0, \tag{2.14e}$$

$$\frac{\partial \rho_s'}{\partial t} = \sigma^{-1}E\delta^{-2}\frac{\partial^2 \rho_s^0}{\partial z^2}. \tag{2.14f}$$

For convenience we introduce a function ϕ defined by

$$\phi = -\partial p^0/\partial t. \tag{2.15}$$

From (2.13) and (2.15) we obtain

$$\partial u^0/\partial t = \partial \phi/\partial y, \tag{2.16a}$$

$$\partial v^0/\partial t = -\partial \phi/\partial x, \tag{2.16b}$$

$$\partial \rho^0/\partial t = \partial \phi/\partial z, \tag{2.16c}$$

$$w^0 = 0, \tag{2.16d}$$

$$\rho_s^0 = \rho_{s(t=0)}^0. \tag{2.16e}$$

Introducing (2.16) into (2.14*a*, *b*, *d*) we obtain

$$u' = \frac{\partial \phi}{\partial x}-\frac{\partial p'}{\partial y}, \tag{2.17a}$$

$$v' = \frac{\partial \phi}{\partial y}+\frac{\partial p'}{\partial x}, \tag{2.17b}$$

$$w' = \left(-\frac{\partial \rho^0}{\partial z}B^2\right)^{-1}\frac{\partial \phi}{\partial z}. \tag{2.17c}$$

Equation (2.17) combined with the first-order continuity equation (2.14*e*) gives

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\left(-\frac{\partial \rho_s^0}{\partial z}B^2\right)^{-1}\frac{\partial^2 \phi}{\partial z^2} = 0. \tag{2.18}$$

Equation (2.18) is a linearized version of the 'quasigeostrophic potential vorticity equation' (see, for example, Charney 1949 or Phillips 1963).

The counterparts of (2.16), (2.17) and (2.18) governing axisymmetric processes are obtained with a similar procedure. With $u = u^0+\delta u'\ldots$ as the radial, $v = v^0+\delta v'\ldots$ as the azimuthal and $w = w^0+\delta w'\ldots$ as the vertical non-dimensional velocity in the polar co-ordinate system (r, φ, z) we obtain, utilizing conditions (2.9*a*, *b*, *d*, *e*),

$$u^0 = 0, \tag{2.19a}$$

$$\partial v^0/\partial t = -\partial \phi/\partial r, \tag{2.19b}$$

$$\partial \rho^0/\partial t = \partial \phi/\partial z, \tag{2.19c}$$

$$w^0 = 0, \tag{2.19d}$$

$$\rho_s^0 = \rho_{s(t=0)}^0, \tag{2.19e}$$

$$u' = \partial\phi/\partial r, \tag{2.20a}$$

$$w' = \left(-\frac{\partial\rho_s^0}{\partial z}B^2\right)^{-1}\frac{\partial\phi}{\partial z}, \tag{2.20b}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\left(-\frac{\partial\rho_s^0}{\partial z}B^2\right)^{-1}\frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.21}$$

We require that the basic stratification ρ_s is stable, that is

$$\partial\rho_s/\partial z = \partial\rho_s^0/\partial z + O(\delta) < 0,$$

which means that (2.18) and (2.21) are elliptic. In the examples considered below we will assume

$$\partial\rho_s^0/\partial z = (\partial\rho_s^0/\partial z)_{t=0} = -1. \tag{2.22}$$

It is, however, important to notice that the general properties of (2.18) and (2.21) indicate that the qualitative behaviour of the solution will not depend critically on this assumption as long as ρ_s^0 is a smooth function of z.

2.2. *Discussion of initial and boundary conditions*

The expansion (2.12) used to derive (2.16)–(2.21) is based on conditions (2.9) which are only valid in the interior. Accordingly we cannot expect that solutions to (2.16)–(2.21) should describe the behaviour in the diffusive regions close to the boundary. This means that our solution will in general not be able to satisfy the complete boundary conditions prescribed at the wall. Similarly, the condition on the timescale given by (2.9*b*) indicates that the initial conditions must be chosen within certain restrictions.

Let us first examine what we must specify to determine a solution from (2.16)–(2.18) or (2.19)–(2.21). The purpose of the analyses is to describe the development in time of the zero-order fields $(\mathbf{v}^0, \rho^0, \rho_s^0)$. From (2.16) or (2.19) we find that if ϕ is known these zero-order fields are completely determined from the initial distributions $(\mathbf{v}^0, \rho^0, \rho_s^0)_{t=0}$. In particular we observe that the zero-order basic density field ρ_s^0 is independent of time whether $\rho_{s(t=0)}^0$ is a linear function of z or not, and that the boundary condition on the density anomaly (that is in general the thermal boundary condition) does not influence ρ_s^0.

The determination of ϕ on the other hand requires only specification of a boundary condition sufficient to solve the elliptic equations (2.18) or (2.21). This boundary condition may be formulated in terms of $(\mathbf{v}.\mathbf{n})_B$, the interior velocity normal to the boundary evaluated at the boundary. In the axisymmetric case or when the boundary is horizontal, this is a straightforward matter using (2.20) and (2.19*a*, *d*) or (2.17*c*) and (2.16*d*) respectively. A non-horizontal boundary in the three-dimensional case requires a somewhat more detailed examination. Thus (2.16) provides a relation between the variation of ϕ along a horizontal tangent to the boundary and $(\mathbf{v}^0.\mathbf{n})_B$ while (2.17) gives a condition on ϕ in terms of the total outflow from the boundary at a certain level. When combined these conditions should be sufficient to determine ϕ. This case will, however, not appear in the examples we are going to analyze.

To sum up we find that the solution in the interior is determined if $(\mathbf{v}^0, \rho^0, \rho_s^0)_{t=0}$ and $(\mathbf{v}.\mathbf{n})_B$ are specified.

The physical counterpart of $(\mathbf{v}.\mathbf{n})_B$ is the velocity perpendicular to the boundary just outside the diffusive boundary layer attached to the boundary. Accordingly $(\mathbf{v}.\mathbf{n})_B$ is not necessarily given by the normal velocity at the boundary; we must take convergence in the boundary-layer transport into account. A general treatment of the boundary layers occurring in rotating inhomogeneous fluid flow will be presented by the author in the near future. In this connexion we will only apply a few simple special cases, which can also be found in the paper by Pedlosky (1967). As a background we will, however, state briefly some general properties of the boundary-layer flow.

The transport, which is the quantity of interest in this connexion, depends on the degree to which the interior fields do not satisfy the real boundary conditions on the density anomaly and the velocity. Observe that the boundary condition on the total density anomaly (in the interior represented by $\rho_s^* + \rho^* = Q_s \rho_s + Q\rho$) should be considered, not only the deviations ρ^* from the basic field ρ_s^*. It turns out (Walin, in preparation) that the behaviour of the boundary layer depends critically on the magnitude of B and $|\tan\varphi|$ where φ is the angle between $\mathbf{n}$ and $\mathbf{k}$. When B or $|\tan\varphi|$ increases, that is when the boundary becomes steeper or the basic stratification stronger, the thermal conditions become successively more important. When $B|\tan\varphi| \ll 1$, the kinematical conditions dominate, and the Ekman layer theory for a homogeneous fluid may be used to determine the boundary-layer transport. On the other hand, when $B|\tan\varphi| \gg 1$ the transport crossing horizontal surfaces is completely determined by the interior field of density anomaly, $\rho_s^* + \rho^*$, and the associated boundary condition at the wall. Furthermore, if the boundary is insulated this transport is $O(E)$ which is negligible compared to the Ekman transport on a rigid boundary $O(E^{\frac{1}{2}})$, since $E^{\frac{1}{2}} \ll 1$. Analogously for a stress free horizontal ($B|\tan\varphi| \ll 1$) boundary the boundary-layer transport becomes $O(E)$. When B and $|\tan\varphi|$ are $O(1)$ the thermal and kinematical influence on the boundary-layer transport cannot be separated and this intermediate case will be avoided in the present context. Since we want to concentrate on mechanically driven flow, we will assume that the boundaries are either horizontal, in which case the boundary-layer transport is obtained from the Ekman theory, or vertical and insulated in which case the vertical transport is $O(E)$. It may be pointed out that thermally driven flow may be considered separately by assuming that the horizontal boundaries are stress free, while the vertical boundaries are subject to thermal forcing.

3. Source flow in unbounded region

We will consider the process that results from the introduction of fluid inside a finite region of an unbounded rotating fluid with a stratification of the form (2.22). Specifically we want to describe the distribution of the swirling flow that will be accelerated in the surroundings of the source. The source region, located in the proximity of a point O at $r = z = 0$, is assumed to be cylindrically symmetric around the z-axis. The integrated volume flux through any surface S enclosing the source region is $Ff(t)$ where $f(t) = O(1)$ and F is the strength of the source (with dimension 'volume/time'). In terms of the non-dimensional velocity $\mathbf{v}$ we obtain the integral constraint

$$UL^2 \iint_S \mathbf{v}_s . \mathbf{n}\, dS = Ff(t), \tag{3.1}$$

where $\mathbf{v}_s$ represents $\mathbf{v}$ evaluated on the surface S enclosing the source and dS is a surface element on S with local unit normal $\mathbf{n}$.

Sufficiently far from O, say outside S_0, the perturbations generated by the source necessarily become sufficiently small for (2.19)–(2.21) to be applicable. Outside S_0 (3.1) may be expressed in terms of ϕ using (2.12), (2.19) and (2.20). If the surface is cylindrically symmetric we obtain to lowest order in δ

$$UL^2\delta \iint_S \left(n_r \frac{\partial \phi}{\partial r} + n_z B^{-2} \frac{\partial \phi}{\partial z}\right) dS = Ff(t), \tag{3.2}$$

where n_r and n_z are the radial and vertical components of $\mathbf{n}$, and S is assumed to lie outside S_0.

Equation (2.21) subject to (3.2) has the solution

$$\phi = \phi_1 + \phi_2, \tag{3.3a}$$

where
$$\phi_1 = -\frac{F}{UL^2} \delta^{-1} f(t) \frac{B}{4\pi} D^{-1}, \tag{3.3b}$$

$$\phi_2 = g(r,z,t)\, D^{-2} \tag{3.3c}$$

and
$$D^2 = r^2 + (Bz)^2. \tag{3.3d}$$

The meridional circulation associated with ϕ_2 does not contribute to the volume flux through S. If the length scale of S_0, outside which (3.3) is assumed to be valid, is chosen as reference scale, we have

$$D \leqslant O(1) \quad \text{outside } S_0.$$

Since we must require
$$|\phi| \leqslant O(1),$$

we obtain the two conditions (remembering that $B = O(1)$)

$$F/UL^2 \leqslant O(\delta) \tag{3.4}$$

and
$$g(r,z,t) \leqslant O(1). \tag{3.5}$$

Using (2.8a, b), (3.4) may be written

$$F\tau/L^3 \leqslant O(R)$$

or, in view of (2.9d),
$$F\tau/L^3 \ll 1. \tag{3.6}$$

The unspecified function $g(r,z,t)$ depends on the distribution over S_0 of $\mathbf{v}_{S_0} . \mathbf{n}$. Let us assume that $\mathbf{v}_{S_0} . \mathbf{n}$ is a smooth function satisfying

$$\iint_{S_0} |\mathbf{v}_{S_0} . \mathbf{n}|\, dS_0 = O\left(\iint_{S_0} \mathbf{v}_{S_0} . \mathbf{n}\, dS_0\right). \tag{3.7}$$

In this case we have
$$\phi_2 = O(\phi_1) \quad \text{on} \quad S_0 \tag{3.8}$$

and (3.5) is automatically satisfied if (3.4) is valid. Equations (3.8) and (3.3) imply

$$\phi_2 = D^{-1}O(\phi_1).$$

Accordingly, ϕ is dominated by ϕ_1 when $D \gg 1$.

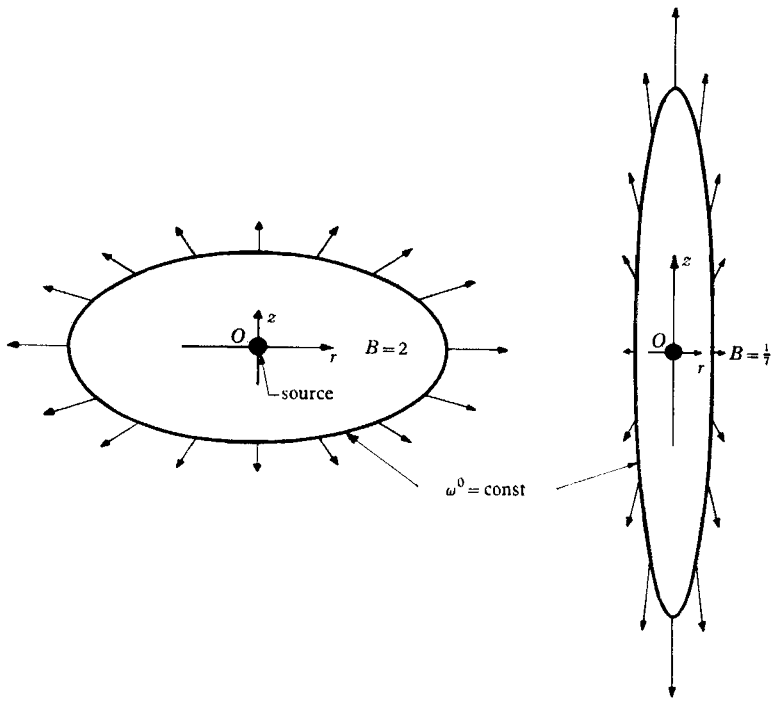

FIGURE 1. Source flow in stratified rotating fluid. Illustration of meridional flow and distribution of angular velocity (ω^0) with dominating ($B = 2$) and weak ($B = \frac{1}{7}$) stratification.

The meridional velocities associated with ϕ_1 are obtained from (3.3*b*) using (2.12), (2.19) and (2.20). To lowest order in δ we obtain

$$u = \delta\frac{\partial\phi}{\partial r} = \frac{F}{UL^2}f(t)\frac{B}{4\pi}\frac{r}{D^3}, \tag{3.9a}$$

$$w = \delta B^{-2}\frac{\partial\phi}{\partial z} = \frac{F}{UL^2}f(t)\frac{B}{4\pi}\frac{z}{D^3}. \tag{3.9b}$$

The meridional flow given by (3.9) is illustrated in figure 1. We observe that when $B = 1$, (3.9) is identical with the corresponding expressions for source flow in a non-rotating homogeneous fluid. When $B > 1$ the outflow occurs mainly in horizontal directions, while when $B < 1$ the flow is concentrated around the z-axis.

The zero-order swirling flow v^0 connected with ϕ_1 is obtained from (3.3) with the aid of (2.19*b*)

$$v^0 = \frac{F}{UL^2}\delta^{-1}\int_0^t f(t)\,dt\frac{B}{4\pi}\frac{r}{D^3} + v^0_{(t=0)}. \tag{3.10}$$

We must require $|v^0| \leqslant O(1)$, which in view of (3.4) gives the additional condition

$$\left|\int_0^t f(t)\,dt\right| \leqslant O(1). \tag{3.11}$$

Equation (3.11) implies that

$$V = O(F\tau), \tag{3.12}$$

where V is the total volume introduced in or sucked out from the fluid region by the source (or sink). Equations (3.6) and (3.11) may be expressed as

$$V \ll V_0, \tag{3.13}$$

where $V_0 \sim L^3$ is the volume enclosed by S_0. Equations (3.11) and (3.13) reflect the fact that a stationary source creates a steady acceleration of the fluid. Accordingly if the solution should be valid outside a fixed surface S_0 we must restrict the total amount of fluid introduced by the source.

The quantity easiest to observe in an experiment is probably the angular velocity $\omega^0 = v^0/r$. From (3.10) we obtain

$$\omega^0 = \frac{F}{UL^2}\delta^{-1}\int_0^t f(t)\,dt\,\frac{B}{4\pi}D^{-3} + \omega^0_{(t=0)}. \tag{3.14}$$

If $\omega^0_{(t=0)} = 0$ the angular velocity is constant on the ellipsoidal surfaces

$$D^2 = r^2 + (Bz)^2 = \text{constant}$$

as illustrated in figure 1.

4. Decay of motion in a horizontally unbounded region

Let us study the time-development of perturbations satisfying (2.9) on a rotating stratified fluid confined between two horizontal rigid surfaces at $z = \pm 1$. The boundaries are at rest in the rotating reference frame and the initial basic density field is of the form (2.22). We will assume that the lateral boundaries are sufficiently far away to be neglected, which requires

$$\iint_{-\infty}^{\infty} |\psi|^2\,dt\,dy \leqslant M, \tag{4.1}$$

where ψ is any one of the perturbation fields $\mathbf{v}$, ρ and p, and M is a finite constant.

As indicated in § 2.2 and also correctly assumed by Holton (1965) the boundary condition at $z = \pm 1$ is that given by the Ekman theory for homogeneous fluids as derived, for example, by Charney & Eliassen (1949). In terms of the interior non-dimensional velocity field, the boundary condition takes the form

$$w = \mp \tfrac{1}{2}\sqrt{2}\,E^{\frac{1}{2}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + O(E) \quad \text{at} \quad z = \pm 1. \tag{4.2}$$

The 'interior' vertical velocity w in (4.2), physically corresponds to the vertical velocity just outside the thin boundary layers (of non-dimensional thickness $E^{\frac{1}{2}}$), and is caused by the spatial variations in the transport carried by the boundary layers. According to (2.12) and (2.16) we have $w = O(\delta)$. If, as in this case, the system is driven by the Ekman layers, the scale of w introduced through (4.2) will characterize the process which suggests the choice,

$$\delta = E^{\frac{1}{2}}. \tag{4.3}$$

Introducing (4.3) into (4.2) and expanding according to (2.12), the lowest order equation becomes

$$B^{-2}\frac{\partial \phi}{\partial z} = \mp \tfrac{1}{2}\sqrt{2}\left(\frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}\right) \quad \text{at} \quad z = \pm 1, \tag{4.4}$$

where we have used (2.17c) for the elimination of w'. Differentiating with respect to time and using (2.16a, b) to eliminate u^0 and v^0 in (4.4) we obtain

$$B^{-2}\frac{\partial^2 \phi}{\partial t\, \partial z} = \pm \tfrac{1}{2}\sqrt{2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \quad \text{at} \quad z = \pm 1. \tag{4.5}$$

If (4.1) holds a general solution to (2.18) satisfying the boundary conditions at $z = \pm 1$ in the form (4.5) may be written as the Fourier integral

$$\phi = \iint_{-\infty}^{\infty} (A_1(k,l)f + A_2(k,l)g) \exp i(kx+ly)\, dk\, dl, \tag{4.6a}$$

where

$$f = \exp(-qt) \cosh mz / \cosh m, \tag{4.6b}$$

$$g = \exp(-rt) \sinh mz / \sinh m, \tag{4.6c}$$

$$m^2 = B^2(k^2+l^2) = B^2 h^2, \tag{4.6d}$$

$$q = \tfrac{1}{2}\sqrt{2}\, B\sqrt{(k^2+l^2)} \cosh m / \sinh m, \tag{4.6e}$$

$$r = \tfrac{1}{2}\sqrt{2}\, B\sqrt{(k^2+l^2)} \sinh m / \cosh m, \tag{4.6f}$$

and the spectral distributions $A_1(k,l)$ and $A_2(k,l)$ are at our disposal to satisfy the initial conditions. The zero-order horizontal velocity field (u^0, v^0) associated with (4.6) is obtained from (2.16a, b).

$$u^0 = u^0_d + u^0_\infty, \tag{4.7a}$$

$$v^0 = v^0_d + v^0_\infty, \tag{4.7b}$$

where the first terms decay with time:

$$u^0_d = \iint_{-\infty}^{\infty} -il\left(\frac{A_1}{q}f + \frac{A_2}{r}g\right) \exp i(kx+ly)\, dk\, dl, \tag{4.7c}$$

$$v^0_d = \iint_{-\infty}^{\infty} -ik\left(\frac{A_1}{q}f + \frac{A_2}{r}g\right) \exp i(kx+ly)\, dk\, dl, \tag{4.7d}$$

while the functions of integration u^0_∞ and v^0_∞ are independent of time. We have

$$(u^0, v^0)_{t\to\infty} = (u^0_\infty, v^0_\infty). \tag{4.8}$$

(u^0_∞, v^0_∞) is the quasistationary velocity field remaining when the process we are studying here has terminated. In a study of the final decay of the velocity field, characterized by $\delta = O(E)$, $(u, v) = (u^0_\infty, v^0_\infty)$ should be used as the initial condition for the horizontal velocity.

The boundary condition in the form (4.4) applied at $t \to \infty$ implies

$$\left(\frac{\partial v^0_\infty}{\partial x} - \frac{\partial u^0_\infty}{\partial y}\right)_{z=\pm 1} = 0. \tag{4.9}$$

From (2.13) it is easily shown that

$$\frac{\partial u^0}{\partial x}+\frac{\partial v^0}{\partial y}=0. \tag{4.10}$$

Equations (4.9) and (4.10) combined with (4.1) require that

$$(u^0_\infty, v^0_\infty)_{z=\pm 1}=0. \tag{4.11}$$

The solution for (u^0, v^0) given by (4.7) may be adjusted to any smooth initial distribution $(u^0, v^0)_{t=0}$ satisfying (4.1) and (4.10). The decaying part (u^0_d, v^0_d) of the velocity field (u^0, v^0) is determined solely from the initial distribution at the boundaries. Since u^0 and v^0 are coupled through (4.10) and (4.1) we need only discuss, for example, u^0. Suppose C_+ and C_- are the Fourier transforms of $u^0_{(t=0,z=\pm 1)}$, defined by

$$u^0_{(t=0,z=\pm 1)}=\iint_{-\infty}^{\infty} C_\pm \exp i(kx+ly)\,dk\,dl. \tag{4.12}$$

Equation (4.12) combined with (4.7) and (4.11) determines A_1 and A_2 according to

$$-\frac{il}{q}A_1=\tfrac{1}{2}(C_+ + C_-), \tag{4.13a}$$

$$-\frac{il}{r}A_2=\tfrac{1}{2}(C_+ - C_-). \tag{4.13b}$$

When $A_1(k,l)$ and $A_2(k,l)$ have been determined from the initial distribution at $z=\pm 1$ according to (4.13), (u^0_d, v^0_d) is known everywhere and (u^0_∞, v^0_∞) may be determined from the initial condition in the interior of the system.

$$(u^0, v^0)_{t=0}=(u^0_d, v^0_d)_{t=0}+(u^0_\infty, v^0_\infty). \tag{4.14}$$

Introducing (4.13) and (4.14) into (4.7) we obtain the solution corresponding to the initial distribution $(u^0, v^0)_{t=0}$

$$(u^0, v^0)=(u^0_d, v^0_d)+(u^0, v^0)_{t=0}-(u^0_d, v^0_d)_{t=0}, \tag{4.15a}$$

where

$$u^0_d=\iint_{-\infty}^{\infty}[\tfrac{1}{2}(C_+ + C_-)f+\tfrac{1}{2}(C_+ - C_-)g]\exp i(kx+ly)\,dk\,dl, \tag{4.15b}$$

$$v^0_d=\iint_{-\infty}^{\infty}-\frac{k}{l}[\tfrac{1}{2}(C_+ + C_-)j+\tfrac{1}{2}(C_+ - C_-)g]\exp i(kx+ly)\,dk\,dl. \tag{4.15c}$$

The solution (4.15) will not be discussed for any particular choice of $(u^0, v^0)_{t=0}$. However, we may draw some general conclusions regarding (4.15) from the dependence of f and g on the horizontal wave-number $h=(k^2+l^2)^{\frac{1}{2}}$.

Suppose the initial distribution is barotropic and dominated by energy in horizontal wave-numbers small compared to B^{-1}. When the initial distribution is barotropic we have $C_+ = C_-$ and we need only discuss the behaviour of f and q, connected with the symmetric part of the solution. For small hB (4.6b, e) may be written

$$f=\exp(-qt)(1+O(hB), \tag{4.16a}$$

$$q=\tfrac{1}{2}\sqrt{2}(1+O(hB)). \tag{4.16b}$$

When $hB \to 0$ (4.16) tend towards the corresponding expressions for the decay process in a homogeneous fluid. Thus when h is small compared with B^{-1} and the initial distribution is barotropic, the influence of the stratification on the decay process is negligible. When the horizontal wave-number is large compared with B^{-1} (4.6b, c, e, f) may be expressed approximately as

$$f = \exp(-qt)[\exp(-mz_1)+\exp(-mz_2)], \tag{4.17a}$$

$$g = \exp(-rt)[\exp(-mz_1)-\exp(-mz_2)], \tag{4.17b}$$

$$q = \tfrac{1}{2}\sqrt{2}\, m = \tfrac{1}{2}\sqrt{2}\, Bh, \tag{4.17c}$$

where $$z_1 = 1-z \quad \text{and} \quad z_2 = 1+z.$$

According to (4.17), f and g are small everywhere except in layers of thickness $m^{-1} = (hB)^{-1}$ near the boundaries. Thus the decay process penetrates only a distance $(hB)^{-1}$ into the fluid from the boundaries. From (4.17c) we may derive a dimensional characteristic timescale, T.

$$T = q^{-1}\tau = \sqrt{2}\, E_D^{-\frac{1}{2}}(2\Omega)^{-1}, \tag{4.18a}$$

where $$E_D = \nu/2\Omega(L/hB)^2 \tag{4.18b}$$

is the Ekman number based on the (dimensional) penetration depth L/hB associated with a perturbation with horizontal scale L/h. Thus the decay time may be considered as the 'spin up' time based on the real penetration depth instead of the depth of the fluid system. It may also be of interest to note that the horizontal scale BH associated with a given penetration depth H is closely analogous to a concept more familiar to meteorologists, namely the so-called 'Rossby radius of deformation'.

This qualitative behaviour of the decay time on the parameter B was correctly displayed by Holton (1965). A relation similar to (4.6d) for the penetration height of disturbances driven from the lower boundary of a stable atmosphere was obtained by Rossby (1938).

5. Decaying motion in closed region

We will study the development of axisymmetric perturbations on a rotating stratified fluid confined in the region

$$r < a,$$
$$-1 < z < 1.$$

The containing surface is at rest in the rotating reference frame and the stratification is of the form (2.22). The vertical boundary at $r = a$ is assumed to be insulated. The analysis is very similar to what has been presented in §4, the only essential difference being that we have now a boundary condition on $r = a$ to be considered. In this section we will use the polar co-ordinate system (r, φ, z) and the associated velocity vector (u, v, w) in terms of which (2.19)–(2.21) are expressed.

As in §4, the boundary condition at $z = \pm 1$ should be obtained from the Ekman theory. The total radial transport M_r carried by the Ekman layer is given by

$$M_r = -\tfrac{1}{2}\sqrt{2}\, E^{\frac{1}{2}} 2\pi r v_{(z=\pm 1)} + O(E), \tag{5.1}$$

where v is the interior zonal velocity field. The physical counterpart of $v_{(z=\pm 1)}$ is the zonal velocity just outside the thin boundary layer. If M_r changes with r, fluid must leave the boundary layer, and from elementary considerations (or from (4.2)) we obtain

$$2\pi r w = \mp \tfrac{1}{2}\sqrt{2}\, E^{\frac{1}{2}} \frac{\partial}{\partial r}[2\pi r v_{(z=\pm 1)}] + O(E). \tag{5.2}$$

As shown by Pedlosky (1967) and also by the present writer (see § 2.2) the transport M_z carried upwards by the boundary layer on $r = a$ is given by

$$M_z = O(E). \tag{5.3}$$

In the corners ($r = a$, $z = \pm 1$) the boundary layer transport changes abruptly and a volume flux $O(E^{\frac{1}{2}})$ will have to leave the corner. This will give rise to a singularity in the meridional circulation. It was precisely this possibility of a significant volume flux from the corner that was overlooked by Pedlosky (1967) in his treatment of the 'stratified spin up'. In this problem, as discussed later in this section, the meridional flow involves initially a concentrated outflow from the corner region caused by a jump in the boundary-layer transport. Since our analysis is valid only if we have to deal with smooth fields, such a concentrated outflow cannot be described. Thus we must require

$$v = O(E^{\frac{1}{2}}) \quad \text{at} \quad r = a,\ z = \pm 1. \tag{5.4}$$

From (5.3) we obtain
$$u = O(E) \quad \text{at} \quad r = a. \tag{5.5}$$

For the same reason as in § 4 we choose

$$\delta = E^{\frac{1}{2}} \tag{5.6}$$

and expand (5.2), (5.4) and (5.5) in δ. The resulting lowest order equations become

$$B^{-2}\frac{\partial \phi}{\partial z} = \mp \tfrac{1}{2}\sqrt{2}\,\frac{1}{r}\frac{\partial}{\partial r}(r v^0) \quad \text{at} \quad z = \pm 1, \tag{5.7}$$

$$\partial \phi/\partial r = 0 \quad \text{at} \quad r = a, \tag{5.8}$$

$$v^0 = 0 \quad \text{at} \quad r = a,\ z = \pm 1. \tag{5.9}$$

Eliminating v^0 from (5.7) with the aid of (2.19b) we obtain the alternative form of the boundary condition at $z = \pm 1$.

$$B^{-2}\frac{\partial^2 \phi}{\partial t\,\partial z} = \pm \tfrac{1}{2}\sqrt{2}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) \quad \text{at} \quad z = \pm 1. \tag{5.10}$$

From (2.19) combined with (5.8) we obtain

$$\partial v^0/\partial t = 0 \quad \text{at} \quad r = a.$$

Thus (5.9) is valid for all times if

$$v^0_{(t=0)} = 0 \quad \text{at} \quad r = a, \quad z = \pm 1. \tag{5.11}$$

The implications of (5.11) will be discussed later on in connexion with the so-called 'spin up' problem.

Equation (2.21) subject to the boundary conditions (5.8) and (5.10) has the general solution

$$\phi = \sum_{\nu=1}^{\infty} (A_\nu f_\nu + B_\nu g_\nu)\, J_0(\alpha_\nu r/a), \tag{5.12a}$$

where $$J_1(\alpha_\nu) = 0, \quad \nu = 1, 2, 3 \ldots, \tag{5.12b}$$

$$f_\nu = \exp(-q_\nu t) \cosh m_\nu z / \cosh m_\nu, \tag{5.12c}$$

$$g_\nu = \exp(-r_\nu t) \sinh m_\nu z / \sinh m_\nu, \tag{5.12d}$$

$$m_\nu = B(\alpha_\nu / a), \tag{5.12e}$$

$$q_\nu = \tfrac{1}{2}\sqrt{2}\, B(\alpha_\nu / a) \cosh m_\nu / \sinh m_\nu, \tag{5.12f}$$

$$r_\nu = \tfrac{1}{2}\sqrt{2}\, B(\alpha_\nu / a) \sinh m_\nu / \cosh m_\nu. \tag{5.12g}$$

The zonal velocity field associated with (5.12) is obtained with the aid of (2.19*b*)

$$v^0 = v_d^0 + v_\infty^0(r, z), \tag{5.13a}$$

where $$v_d^0 = -\sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{a} \left(\frac{A_\nu}{q_\nu} f_\nu + \frac{B_\nu}{r_\nu} g_\nu \right) J_1(\alpha_\nu r / a). \tag{5.13b}$$

Since $\phi(t \to \infty) = 0$ we obtain from (5.7) combined with (5.9)

$$v^0_{\infty(z=\pm 1)} = 0. \tag{5.14}$$

We observe that (5.12*b*) implies

$$v^0_{d(r=a)} = 0. \tag{5.15}$$

Equation (5.13) may be adjusted to any smooth initial distribution $v^0_{(t=0)}$ satisfying (5.11). As in §4 we find that v_d^0 is determined by the initial condition on the horizontal boundaries.

Let us expand $v^0_{(t=0, z=\pm 1)}$ in the first-order Bessel functions according to

$$v^0_{(t=0, z=\pm 1)} = \sum_{\nu=1}^{\infty} C_{\nu\pm} J_1(\alpha_\nu r / a), \tag{5.16}$$

where (5.16) combined with (5.13) determines A_ν and B_ν whereupon v_∞^0 is determined from

$$v^0_{(t=0)} = v^0_{d(t=0)} + v^0_\infty$$

and we obtain $$v^0 = v_d^0 + v^0_{(t=0)} - v^0_{d(t=0)}, \tag{5.17a}$$

where $$v_d^0 = \sum_{\nu=1}^{\infty} \left(\tfrac{1}{2}(C_{\nu+} + C_{\nu-}) f_\nu + \tfrac{1}{2}(C_{\nu+} - C_{\nu-}) g_\nu \right) J_1(\alpha_\nu r / a). \tag{5.17b}$$

Let us now discuss the implications of condition (5.11) that we have been forced to impose on the initial condition for v^0. Suppose that the initial distribution of zonal velocity, outside the thin boundary layers at the walls, is given by $F(r, z)$ with the property

$$F(r, z) = O(1) \quad \text{at} \quad z = \pm 1, \quad r = a. \tag{5.18}$$

In view of (5.11), we are not allowed to use $F(r, z)$ as initial condition for v^0. The reason for this failure is that v^0 only describes the development of the zonal velocity in a certain time range defined by

$$t^* = O(\tau), \tag{5.19a}$$

where $$(2\Omega)^{-1} \ll \tau \ll L^2/\nu. \tag{5.19b}$$

In (5.19), which is a consequence of (2.9*b*), t^* represents dimensional time $t^* = \tau t$. Thus any changes of the zonal velocity field, occurring on a time scale small compared to τ, must be incorporated in the initial condition for v^0. In space we have a corresponding phenomena represented by the boundary layer, not described by v^0, but taken into account through a modification of the boundary condition. According to (5.1) and (5.3), we obtain a discontinuity in the boundary-layer transport, if the initial zonal velocity field is given by $F(r,z)$. As a consequence we obtain a concentrated meridional circulation close to the corner. This

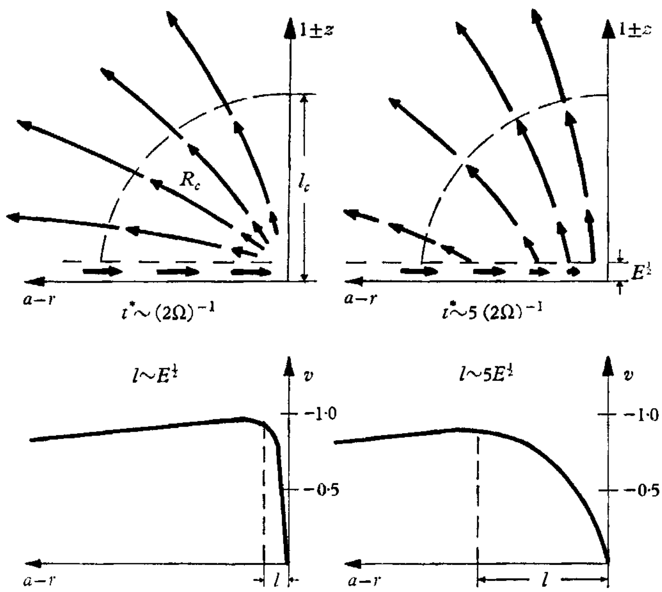

FIGURE 2. The corner regions in 'stratified spin up'. Illustration of the zonal velocity (v) just outside the Ekman layers and the associated meridional circulation (upper part of the figure) at $t^* \sim (2\Omega)^{-1}$ (to the left) and $t^* \sim 5(2\Omega)^{-1}$ (to the right). The width l of the region where the boundary-layer flux leaves the boundary layer, grows at least as $l \sim E^{\frac{1}{2}}\, t^*/(2\Omega)^{-1}$. Thus when $t^* \sim \tau$ we have $l \gg E^{\frac{1}{2}}$, since $\tau \gg (2\Omega)^{-1}$. The thick arrows indicate the transport carried by the boundary layer.

strong meridional flow will necessarily cause a development close to the corner with time scale small compared to τ which should be taken into account when deriving the initial condition for v^0. Because of analytical difficulties we are unable to describe this development in detail. However, it is quite evident that the meridional circulation will tend to decrease the zonal velocity above the boundary layer, thereby eliminating the discontinuity in the boundary-layer transport.

A qualitative picture of the development close to the corners is displayed in figure 2. Essentially figure 2 is based on the assumption that the rate of decay of the zonal velocity is roughly proportional to the intensity of the meridional circulation. In reality the decay may be somewhat faster because of non-linear effects etc.

The development in the corner region, although not known in detail, may be taken into account in a qualitative way if we assume

$$v^0_{(t=0)} = \begin{cases} F(r,z) & \text{outside} \quad R_c, \\ 0 & \text{at} \quad r=a, \quad z=\pm 1, \end{cases} \tag{5.20}$$

where R_c includes two small regions surrounding the corners ($r = a$, $z = \pm 1$). The non-dimensional length scale l_c of R_c must be chosen sufficiently large for conditions (2.9) to hold when L is replaced by $l_c L$ in the definition of R and E. Obviously we cannot predict the development of v^0 in the corner regions with (5.20) as initial condition. However, the amplitude C_ν for $\nu < l_c^{-1}$

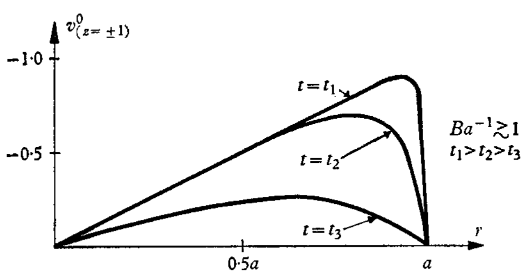

FIGURE 3. The qualitative behaviour of $v^0_{(z=\pm 1)}$ in stratified spin up at three stages in the development. Since the high wave-numbers have a higher rate of decay (see (5.12f)), the distribution becomes smoother when t increases.

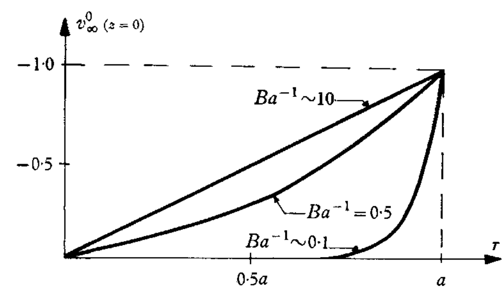

FIGURE 4. The interior zonal velocity at mid depth remaining as $t \to \infty$, $v^0_{\infty(z=0)}$, as a function of r for different values of Ba^{-1}. Since the function v^0_∞ does not include the boundary layer at $r = a$ the viscous boundary condition at $r = a$ is not satisfied by v^0_∞. The real zonal velocity will drop to zero when approaching $r = a$ within a region of dimensional thickness $O(E^{\frac{1}{4}})\,L$. Observe that $v^0_{\infty(z=0)} \simeq v^0_{(t=0)}$ when Ba^{-1} is large, while $v^0_{\infty(z=0)} \simeq 0$ everywhere except close to $r = a$ when Ba^{-1} is small.

is almost independent of the behaviour of $v^0_{(t=0)}$ in R_c and the solution outside R_c is dominated by these low wave-numbers. Thus the behaviour outside R_c will in general be well described if we simply cut off the series at $\nu = N < l_c^{-1}$.

According to (5.20) the so-called 'spin up' problem gives rise to the initial condition

$$v^0_{(t=0)} = \left\{\begin{matrix} -r/a & \text{outside} \quad R_c, \\ 0 & \text{at} \quad r = a, \quad z = \pm 1. \end{matrix}\right\} \tag{5.21}$$

For $\nu \leqslant N < l_c^{-1}$ the coefficients in (5.16) become

$$C_{\nu+} \doteq C_{\nu-} = -2/\alpha_\nu J_2(\alpha_\nu). \tag{5.22}$$

Equation (5.22) inserted in (5.17) gives us the solution

$$v^0 = -\frac{r}{a} - \sum_1^N - \frac{2}{\alpha_\nu J_2(\alpha_\nu)}\,(1 - \exp(-q_\nu t))\,\frac{\cosh m_\nu z}{\cosh m_\nu}\,J_1(\alpha_\nu r/a), \tag{5.23}$$

which we expect to describe the zonal velocity outside R_c in the time range $t = O(1)$.

The qualitative features of (5.23) are illustrated in figures 3, 4 and 5. We observe, that when Ba^{-1} is large compared with 1, the spin up process only penetrates to the dimensional height $B^{-1}aL$ (aL = radius of container). When Ba^{-1} is small compared with 1, the process is essentially identical to the spin up

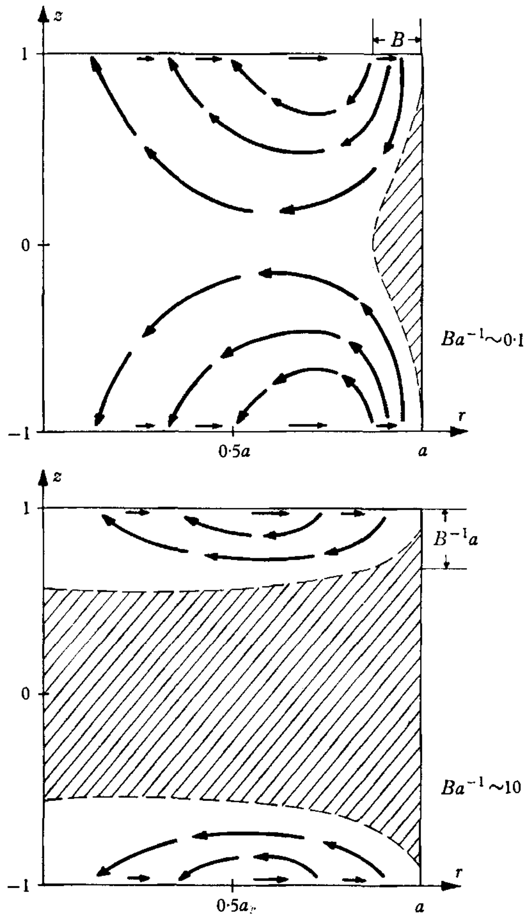

FIGURE 5. Illustration of the meridional circulation (for $t \sim 1$) with dominating ($Ba^{-1} \sim 10$) and weak stratification ($Ba^{-1} \sim 0{\cdot}1$). The shaded areas are essentially untouched by the decay process which is most effective in the centre of the container.

of a homogeneous fluid except in a region close to the vertical boundary of thickness BL, where L is the depth of the fluid region. The boundary layer at $r = a$ is not included in our description of the zonal velocity field. The thickness of this layer is $O(E^{\frac{1}{4}})L$. Thus when $B = O(E^{\frac{1}{4}})$ the small region unaffected by the spin up will disappear and the process becomes identical to the homogeneous spin up. We must observe, however, that the description given by (5.23) is poor in this limit, particularly in the corner regions, since the initially jet-like character of the meridional flow is not effectively smoothed out when Ba^{-1} is small.

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